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A 5-r UNIQUENESS THEOREM

BY

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ABSTRACT. A Borel-regular Carathéodory outer measure Λ on a separable metric space X, where Λ is invariant with respect to a family H of homeomorphisms from X onto X, is unique if Λ satisfies a 5-r condition at one point in X and if H satisfies Condition I, a condition much weaker than, but related to, the invariance of distance under H.

0. **Introduction.** The main result of this paper is a uniqueness theorem for an outer measure on a separable metric space that is invariant with respect to a family of homeomorphisms of the space. Sufficient conditions include a Condition (I) on the family of homeomorphisms and a condition (5-r) on the invariant measure. In [2], Mickle and Radó proved a uniqueness theorem with similar conditions; here the conditions on the invariant measure are weaker. I am grateful to Professor Earl J. Mickle, not only for suggesting this topic for research, but also for his encouragement and help.

In §1 are necessary preliminaries and the statement of the main theorem. In §2 it is shown that if a Haar measure satisfies the 5-r condition at one point, it does so at every point of the space. §3 includes needed density and covering lemmas, §4 contains the proof of the main theorem, and in §5 several other criteria for uniqueness are given.

1. **Preliminaries and uniqueness theorem.** Several definitions will be needed for the development of the theory. In what follows, $c(x, r)$ is the closed sphere of center x and radius r. Throughout, unless otherwise indicated, unions and summations are taken from one to infinity.

DEFINITION 1.1. A real-valued set function Λ, defined on all subsets of a separable metric space (X, d), is said to be a Carathéodory outer measure if

(i) for $E \subset X$, $0 \leqslant \Lambda(E) \leqslant \infty$,

(ii) $\Lambda(\emptyset) = 0$,

(iii) if $E_1 \subset E_2$, then $\Lambda(E_1) \leqslant \Lambda(E_2)$,

(iv) if $E_1, E_2, \cdots$, is any sequence of subsets of X, then $\Lambda(\bigcup E_n) \leqslant \Sigma\Lambda(E_n)$, and

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(v) if E_1 and E_2 are a positive distance apart, then $\Lambda(E_1 \cup E_2) = \Lambda(E_1) + \Lambda(E_2)$.

DEFINITION 1.2. Let (X, d) be a separable metric space, and let H be a family of homeomorphisms from X onto X. A Borel-regular Carathéodory outer measure Λ on X will be called H-invariant, or Haar with respect to H, if $\Lambda[h(E)] = \Lambda(E)$ for every $E \subset X$ and every h in H, and if there is a nonempty set O for which $0 < \Lambda(O) < \infty$.

DEFINITION 1.3. Let (X, d) be a separable metric space, and let H be a family of homeomorphisms from X onto X. The family H is said to satisfy Condition (I) if for every pair of elements x, y in X there are homeomorphisms h_1 and h_2 in H with $h_1(x) = y$ and $h_2(y) = x$, and positive real numbers $l(x, y)$ and $L(x, y)$ such that if $0 < r < l$, then

$$c(x, r) \subset h_2[c(y, rL)] \subset c(x, rL^2), \quad \text{and} \quad c(y, r) \subset h_1[c(x, rL)] \subset c(y, rL^2).$$

DEFINITION 1.4. A Carathéodory outer measure σ on a separable metric space (X, d) is said to satisfy the strong 5-r condition if for every bounded set E there are positive real numbers $k(E)$ and $K(E)$ such that for every x in E if $0 < r < k(E)$, then

$$\sigma[c(x, 5r)] < K(E)\sigma[c(x, r)].$$

DEFINITION 1.5. A Carathéodory outer measure σ on a separable metric space (X, d) is said to satisfy the 5-r condition at x if there are positive constants $k(x)$ and $K(x)$ such that for $0 < r < k(x)$,

$$\sigma[c(x, 5r)] < K(x)\sigma[c(x, r)].$$

If σ satisfies the 5-r condition at every x in X, it is said to satisfy the 5-r condition on X, and it will be called a 5-r measure.

DEFINITION 1.6. A Borel-regular Carathéodory outer measure Λ on X will be called locally finite if for each x in X there is an open set O_x such that $x \in O_x$ and $\Lambda(O_x) < \infty$.

If Λ is locally finite, then, since X is separable, there is a sequence of open sets $O_1, O_2, \cdots$ such that

$$(1) \qquad X = \bigcup O_n \quad \text{and} \quad \Lambda(O_n) < \infty, \qquad n = 1, 2, \cdots.$$

It is clear that if a measure satisfies the strong 5-r condition, it satisfies the 5-r condition. Note also that if Λ satisfies the 5-r condition at x, there is an open set containing x that has finite Λ-measure, hence if Λ satisfies the 5-r condition on X, then Λ is locally finite.

The main uniqueness theorem which will be proved in this paper can now be stated.

THEOREM 1.7. (UNIQUENESS THEOREM). *Let (X, d) be a separable metric space, let H be a family of homeomorphisms from X onto X which satisfies Condtion* (I), *and let Λ_1 and Λ_2 be Borel-regular Carathéodory outer measures on X that are Haar with respect to H. Then if Λ_1 satisfies the 5-r condition at one point in X, there is a positive real number c such that for every $E \subset X$, $\Lambda_2(E) = c\Lambda_1(E)$.*

Theorem 1.7 is a sharpening of the 5-r uniqueness theorem proved by Mickle and Radó [2] in 1959. Their theorem established the uniqueness of an invariant measure on the space of oriented lines in three-space, a result which does not follow from the standard uniqueness theorems for invariant measures. In their original theorem, one of the invariant measures is required to satisfy the strong 5-r condition. In the present theorem, this condition is weakened to the requirement that one of the invariant measures satisfy the 5-r condition at one point.

2. One-point theorem.

THEOREM 2.1. *Let (X, d) be a separable metric space, let H be a family of homeomorphisms from X onto X satisfying Condition* (I), *and let Λ be an H-invariant outer measure on X that satisfies the 5-r condition at one point. Then Λ satisfies the 5-r condition everywhere on X.*

PROOF. Assume for $0 < r < k$, $\Lambda[c(x_0, 5r)] < K\, \Lambda[c(x_0, r)]$, and let x be an arbitrary element in X. Then, since H satisfies Condition (I), there are positive constants $l_{(x_0,x)}, L_{(x_0,x)}$, and h, h_0 in H with $h(x) = x_0$ and $h_0(x_0) = x$ such that if $0 < r < l$,

$$c(x, r) \subset h_0[c(x_0, rL)] \subset c(x, rL^2) \quad \text{and} \quad c(x_0, r) \subset h[c(x, rL)] \subset c(x_0, rL^2).$$

Then for $r < (1/25L)\min(k,l)$, we have

$$\Lambda[c(x, 5r)] = \Lambda[h(c(x, 5r))] \leqslant \Lambda[c(x_0, 5rL)],$$

and, letting n be the positive integer with $5^{n-1} \leqslant L < 5^n$,

$$\begin{aligned}\Lambda[c(x_0, 5rL)] &\leqslant \Lambda[c(x_0, r5^{n+1})] < K^{2n+1}\Lambda[c(x_0, r5^{-n})]\\ &\leqslant K^{2n+1}\Lambda[c(x_0, rL^{-1})] = K^{2n+1}\Lambda[h_0(c(x_0, rL^{-1}))]\\ &\leqslant K^{2n+1}\Lambda[c(x, r)],\end{aligned}$$

and Λ satisfies the 5-r condition at an arbitrary x in X, hence on X.

3. **Density and covering lemmas.** There are several different conditions that are sufficient for a Haar measure to satisfy the 5-r condition at a point, and they all concern an underlying Carathéodory outer measure which satisfies the 5-r condition, but which is not necessarily Haar itself. In what follows, this underlying measure will usually be denoted σ, and use will be made of the upper and lower densities of a Haar measure Λ with respect to the underlying 5-r measure σ.

In Lemmas 3.1 through 3.5 below, (X, d) is a separable metric space, and σ is a Carathéodory outer measure which satisfies the 5-r condition on x. Letting

$$A_n = \{x: x \in X,\ 0 < r < 1/n \Rightarrow \sigma[c(x, 5r)] < n\sigma[c(x, r)]\}, \tag{2}$$

we have $X = \bigcup A_n$ and $A_n \subset A_{n+1}$, $n = 1, 2, \cdots$.

LEMMA 3.1. *Let Λ be a Borel-regular Carathéodory outer measure on X, let t be a positive finite number, and let E be a subset of X. Set*

$$G_t(E) = \left\{x: x \in X,\ \limsup_{r\to 0} \frac{\Lambda[c(x, r) \cap E]}{\sigma[c(x, r)]} > t\right\}.$$

Then for the sets A_n in (2), $\sigma[A_n \cap G_t(E)] < (n/t)\Lambda(E)$.

PROOF. Since

$$\begin{aligned} A_n \cap G_t(E) \subset \bigcup \{c(x, r): x \in A_n \cap G_t(E), r < 1/n,\\ \Lambda[c(x, r) \cap E] > t\sigma[c(x, r)]\}, \end{aligned} \tag{3}$$

by a covering theorem of Mickle and Radó [3, p. 328], there exists a pairwise disjoint sequence of these spheres, $c(x_1, r_1), c(x_2, r_2), \cdots$, such that

$$A_n \cap G_t(E) \subset \bigcup \{c(x_i, 5r_i): i = 1, 2, \cdots\}. \tag{4}$$

From (2), (3), and (4) it follows that

$$\begin{aligned} \sigma[A_n \cap G_t(E)] &\leqslant \sum \sigma[c(x_i, 5r_i)] \leqslant n \sum \sigma[c(x_i, r_i)] \\ &< (n/t) \sum \Lambda[c(x_i, r_i) \cap E] = (n/t)\Lambda\left[\bigcup c(x_i, r_i) \cap E\right] \\ &\leqslant (n/t)\Lambda(E). \end{aligned}$$

LEMMA 3.2. *Let Λ be a locally finite Borel-regular Carathéodory outer measure on X and let E be a Λ-measurable subset of X. Then for $x \in X - E$,*

$$\lim_{r\to 0}\frac{\Lambda[c(x,r)\cap E]}{\sigma[c(x,r)]}=0 \quad \text{a.e. } (\sigma). \tag{5}$$

PROOF. It suffices to show that (5) holds for $\Lambda(E)<\infty$, since letting $E_n = E\cap O_n$, with O_n as in (1), we have

$$\left\{x: x\in X-E, \limsup_{r\to 0}\frac{\Lambda[c(x,r)\cap E]}{\sigma[c(x,r)]}>0\right\}$$
$$\subset \bigcup\left\{x: x\in X-E_n, \limsup_{r\to 0}\frac{\Lambda[c(x,r)\cap E_n]}{\sigma[c(x,r)]}>0\right\},$$

and it suffices to show $\sigma(E_n)=0, n=1,2,\cdots$.

Assume $\Lambda(E)<\infty$. For $0<t<\infty$, let

$$H_t(E) \doteq \left\{x: x\in X-E, \limsup_{r\to 0}\frac{\Lambda[c(x,r)\cap E]}{\sigma[c(x,r)]}>t\right\}.$$

For arbitrary $\epsilon>0$ there is a closed set C such that $C\subset E$ and $\Lambda(E-C)<\epsilon$ [4]. Since C is a closed subset of E, for r sufficiently small and for $x\in X-E$, we have $c(x,r)\cap E = c(x,r)\cap(E-C)$, and for $x\in H_t(E)$,

$$\limsup_{r\to 0}\frac{\Lambda[c(x,r)\cap(E-C)]}{\sigma[c(x,r)]}>t.$$

Hence $H_t(E)\subset G_t(E-C)$ and by Lemma 3.1 for each positive integer n,

$$\sigma[A_n\cap H_t(E)] < \epsilon(n/t).$$

Since $\epsilon>0$ is arbitrary, for each positive integer n and $t>0, \sigma[A_n\cap H_t(E)] = 0$, and by (2), $\sigma[H_t(E)]=0$. Then since

$$\left\{x: x\in X-E, \limsup_{r\to 0}\frac{\Lambda[c(x,r)\cap E]}{\sigma[c(x,r)]}>0\right\} = \bigcup H_{1/m}(E),$$

(5) follows, and the lemma is proved.

LEMMA 3.3. *If* Λ *is a Borel-regular Carathéodory outer measure and* E *is a subset of* X *such that for* $x\in E$

$$\liminf_{r\to 0}\frac{\Lambda[c(x,r)\cap E]}{\sigma[c(x,r)]}=0,$$

then $\Lambda(E)=0$.

PROOF. If suffices to show that $\Lambda(A_n \cap E) = 0$ for A_n as in (2). Since there is a sequence (O_m) of open sets such that $X = \bigcup O_m$, $\sigma(O_m) < \infty$, $m = 1, 2, \cdots$, it suffices to show that $\Lambda(A_n \cap O_m \cap E) = 0$. For $x \in E_{nm} = A_n \cap O_m \cap E$, we have

$$\liminf_{r \to 0} \frac{\Lambda[c(x, r) \cap E_{nm}]}{\sigma[c(x, r)]} = 0.$$

Let $\epsilon > 0$ be given. The family of closed spheres $c(x, r/5)$ for which $x \in E_{nm}$, $r < 1/n$, $c(x, r) \subset O_m$, and $\Lambda[c(x, r) \cap E_{nm}] < \epsilon\sigma[c(x, r)]$ covers E_{nm}. Accordingly (by [3]), there is a pairwise disjoint sequence of closed spheres $c(x_1, r_1/5), c(x_2, r_2/5), \cdots$ such that

$$E_{nm} \subset \bigcup c(x_i, r_i)$$

and

$$\begin{aligned}\Lambda(E_{nm}) &= \Lambda\left[\bigcup c(x_i, r_i) \cap E_{nm}\right] \leqslant \sum \Lambda[c(x_i, r_i) \cap E_{nm}] \\ &< \epsilon \sum \sigma[c(x_i, r_i)] < \epsilon n \sum \sigma[c(x_i, r_i/5)] \\ &= \epsilon n \sigma\left[\bigcup c(x_i, r_i/5)\right] \leqslant \epsilon n \sigma(O_m).\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, n and m are fixed, and $\sigma(O_m) < \infty$, we have $\Lambda(E_{nm}) = 0$.

LEMMA 3.4. *Let* Λ *be a locally finite Borel-regular Carathéodory outer measure and let*

$$E = \left\{x \colon x \in X, \limsup_{r \to 0} \frac{\Lambda[c(x, r)]}{\sigma[c(x, r)]} = \infty\right\}.$$

Then $\sigma(E) = 0$.

PROOF. It suffices to show $\sigma(E_{nm}) = 0$, where E_{nm} is defined as in the proof of the previous lemma.

Let $t > 0$ be given. By Lemma 3.1, $\sigma(E_{nm}) \leqslant \sigma[A_n \cap G_t(O_m)] < (n/t)\Lambda(O_m)$, and since t is arbitrary, $\sigma(E_{nm}) = 0$.

LEMMA 3.5. *Let* Λ *be a locally finite Borel-regular Carathéodory outer measure on* X *and let*

$$E = \left\{x \colon x \in X, \liminf_{r \to 0} \frac{\Lambda[c(x, r)]}{\sigma[c(x, r)]} < \infty, \limsup_{r \to 0} \frac{\Lambda[c(x, r)]}{\sigma[c(x, r)]} = \infty\right\}.$$

Then $\Lambda(E) = 0$.

PROOF. By the previous lemma, we have $\sigma(E) = 0$. Let

$$E_j = \left\{ x\colon x \in E,\ \liminf_{r\to 0} \frac{\Lambda[c(x, r)]}{\sigma[c(x, r)]} < j \right\}, \qquad j = 1, 2, \cdots .$$

To show that $\Lambda(E) = 0$, it suffices to show $\Lambda(E_j) = 0$, for $j = 1, 2, \cdots$, and to show that $\Lambda(E_j) = 0$, it suffices to show that $\Lambda(E_{jn}) = 0$, where $E_{jn} = E_j \cap A_n$, with A_n as in (2). Let $\epsilon > 0$ be given. There is an open set O such that $E_{jn} \subset O$ and $\sigma(O) < \epsilon$. (See Mickle and Radó [3].) The set E_{jn} is covered by the family of closed spheres $c(x, r/5)$ such that $x \in E_{jn}$, $r < 1/n$, $c(x, r) \subset O$, and $\Lambda[c(x, r)] < j\sigma[c(x, r)]$. Hence, by [3] there is a pairwise disjoint sequence of the spheres, $c(x_1, r_1/5), c(x_2, r_2/5), \cdots$, such that $E_{jn} \subset \bigcup c(x_i, r_i)$. Then

$$\begin{aligned}\Lambda(E_{jn}) &\leq \sum \Lambda[c(x_i, r_i)] < j \sum \sigma[c(x_i, r_i)] \\ &< jn \sum \sigma[c(x_i, r_i/5)] = jn\sigma\left[\bigcup c(x_i r_i/5)\right] \\ &\leq jn\sigma(O) < jn\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\Lambda(E_{jn}) = 0$.

It should be noted that Lemmas 3.1 through 3.5 are concerned only with the relationship between two measures on X, and that the family H of homeomorphisms is not involved. The following two lemmas concern properties of covers of X, and are also independent of the family H.

LEMMA 3.6. *Let (X, d) be a separable metric space, Λ a locally finite Borel-regular Carathéodory outer measure on X, and σ a Borel-regular Carathéodory outer measure on X that satisfies the 5-r condition. If B is a Borel set with $\Lambda(B) < \infty$, $\sigma(B) > 0$, and if S is a family of Borel sets covering B with the property that for every x in B there is an S_x in S such that $x \in S_x$ and*

$$\limsup_{r\to 0} \frac{\Lambda[c(x, r) \cap S_x]}{\sigma[c(x, r)]} > 0,$$

then there is an S in S with $\Lambda(B \cap S) > 0$.

PROOF. By Lemma 3.2, for a.e. (σ) x in B,

$$\lim_{r\to 0} \frac{\Lambda[c(x, r) \cap (X - B)]}{\sigma[c(x, r)]} = 0,$$

and since $\sigma(B) > 0$, there is such an x. For that x, its associated $S_x = S$, and any radius r, we have

$$\frac{\Lambda[c(x,r)\cap S]}{\sigma[c(x,r)]} \leqslant \frac{\Lambda[c(x,r)\cap S\cap B]}{\sigma[c(x,r)]} + \frac{\Lambda[c(x,r)\cap S\cap (X-B)]}{\sigma[c(x,r)]}$$
$$\leqslant \frac{\Lambda[c(x,r)\cap S\cap B]}{\sigma[c(x,r)]} + \frac{\Lambda[c(x,r)\cap (X-B)]}{\sigma[c(x,r)]}.$$

Hence

$$\limsup_{r\to 0} \frac{\Lambda[c(x,r)\cap S\cap B]}{\sigma[c(x,r)]} \geqslant \limsup_{r\to 0} \frac{\Lambda[c(x,r)\cap S]}{\sigma[c(x,r)]} > 0,$$

and $\Lambda(S\cap B) > 0$.

LEMMA 3.7. *Let (X, d) be a separable metric space, Λ a locally-finite Borel-regular Carathéodory outer measure, and σ a Borel-regular Carathéodory outer measure that satisfies the 5-r condition. If B is a Borel set, and S is a family of Borel sets such that for each x in B, there is an S_x in S with $x \in S_x$ and*

$$\limsup_{r\to 0} \frac{\Lambda[c(x,r)\cap S_x]}{\sigma[c(x,r)]} > 0,$$

then there is a sequence (S_i), $i = 1, 2, \cdots$, of sets in S with $\sigma(B - \bigcup S_i) = 0$.

PROOF. We may assume $\sigma(B) > 0$, since if not the lemma is trivially true. First assume $\Lambda(B) < \infty$, and let

$$a = \inf\left\{\Lambda\left(B - \bigcup S_i\right) : (S_i) \text{ is a sequence of sets in } S\right\}. \tag{6}$$

Since $\Lambda(B) < \infty, a < \infty$. Note that there is a sequence (S_i^*) with $a = \Lambda(B - \bigcup S_i^*)$: take

$$(S_i^*) = \bigcup_{n=1}^{\infty}\left\{(S_{i,n}) \colon \Lambda\left(B - \bigcup_{i=1}^{\infty} S_{i,n}\right) < a + 1/n\right\}.$$

Assume $\sigma(B - \bigcup S_i^*) > 0$. Then $B - \bigcup S_i^*$ and the family S satisfy the conditions for Lemma 3.6, and there is a set S in S with $\Lambda[(B - \bigcup S_i^*)\cap S] > 0$, and $\Lambda[B - (S \cup \bigcup S_i^*)] < a$, contradicting (6). Hence $\sigma(B - \bigcup S_i^*) = 0$.

For arbitrary B, let $B_n = B \cap O_n$, with O_n as in (1). For each n, $n = 1, 2, \cdots$, we have (S_{ni}), $i = 1, 2, \cdots$, such that $\sigma(B_n - \bigcup_{i=1}^{\infty} S_{ni}) = 0$. Then

$$\sigma\left(B-\bigcup_{n=1}^{\infty}\bigcup_{i=1}^{\infty}S_{ni}\right)\leqslant\sigma\left[\bigcup_{n=1}^{\infty}S\left(B_n-\bigcup_{i=1}^{\infty}S_{ni}\right)\right]\leqslant\sum_{n=1}^{\infty}\sigma\left(B_n-\bigcup_{i=1}^{\infty}S_{ni}\right)=0.$$

4. **Lemmas and proof of main theorem.** Throughout §4, (X, d) is a separable metric space, H is a family of homeomorphisms from X onto X that satisfies Condition (I), Λ and σ are Borel-regular Carathéodory outer measures that are Haar with respect to H, and σ satisfies the 5-r condition.

LEMMA 4.1. *Let $E \subset X$ and let $x \in E$ such that*

$$\limsup_{r\to 0}\frac{\Lambda[c(x, r)\cap E]}{\sigma[c(x, r)]}>0,$$

and let $y \in X$. Then there is an $h \in H$ such that $h(x) = y$ and

$$\limsup_{r\to 0}\frac{\Lambda[c(y, r)\cap h(E)]}{\sigma[c(y, r)]}>0.$$

PROOF. Since H satisfies Condition (I), we have $h \in H$, $l(x, y)$ and $L(x, y)$ such that $h(x) = y$ and if $0 < r < l(x, y)$, $c(y, r) \subset h[c(x, rL)] \subset c(y, rL^2)$. We may assume $L(x, y) > 1$. Since σ satisfies the 5-r condition, there is an n such that $x \in A_n$ and $y \in A_n$, with A_n as in (2).

Also there is a positive integer q such that $5^{q-1} \leqslant L(x, y) < 5^q$. Note that $\sigma[c(x, rL)] \leqslant \sigma[c(x, 5^q r)] < n^{2q}\sigma[c(x, 5^{-q}r)] \leqslant n^{2q}\sigma[c(x, r/L)]$.

Then, for $0 < r < l(x, y)/nL$, we have

$$\begin{aligned}\frac{\Lambda[c(y, r)\cap h(E)]}{\sigma[c(y, r)]}&\geqslant\frac{\Lambda\{h[c(x, r/L)]\cap h(E)\}}{\sigma\{h[c(x, rL)]\}}=\frac{\Lambda[c(x, r/L)\cap E]}{\sigma[c(x, rL)]}\\&=\frac{\Lambda[c(x, r/L)\cap E]}{\sigma[c(x, r/L)]}\,\frac{\sigma[c(x, r/L)]}{\sigma[c(x, rL)]}\\&>\frac{\Lambda[c(x, r/L)\cap E]}{\sigma[c(x, r/L)]}\,\frac{1}{n^{2q}},\end{aligned}$$

and

$$\limsup_{r\to 0}\frac{\Lambda[c(y, r)\cap h(E)]}{\sigma[c(y, r)]}>\frac{1}{n^{2q}}\limsup_{r\to 0}\frac{\Lambda[c(x, r)\cap E]}{\sigma[c(x, r)]}>0.$$

LEMMA 4.2. *Let B be a Borel set in X such that $\Lambda(B) > 0$. Then there exists a sequence (h_i), $i = 1, 2, \cdots$, of homeomorphisms in H such that $\sigma[X-\bigcup h_i(B)] = 0$.*

PROOF. By Lemma 3.3, since $\Lambda(B) > 0$, there is a point $x_0 \in B$ with

$$\limsup_{r\to 0} \frac{\Lambda[c(x_0, r) \cap B]}{\sigma[c(x_0, r)]} > 0.$$

Then, by Lemma 4.1, for any $x \in X$ there is an $h_x \in H$ such that $h_x(x_0) = x$ and

$$\limsup_{r\to 0} \frac{\Lambda[c(x, r) \cap h_x(B)]}{\sigma[c(x, r)]} > 0.$$

Thus X is covered by the Borel sets $h_x(B)$ for $x \in X$ in a manner to satisfy Lemma 3.7, and there is a sequence (h_i), $i = 1, 2, \cdots$, in H such that $\sigma[X - \bigcup h_i(B)] = 0$.

LEMMA 4.3. *Let B be a Borel set in X such that $\sigma(B) > 0$. Then there exists a sequence (h_i), $i = 1, 2, \cdots$, of homeomorphisms in H such that $\sigma[X - \bigcup h_i(B)] = 0$.*

PROOF. Since σ satisfies all the conditions placed upon Λ, Lemma 4.2 holds with $\Lambda = \sigma$.

LEMMA 4.4. *Let B be a Borel set in X. Then $\sigma(B) = 0$ if and only if $\Lambda(B) = 0$.*

PROOF. Assume $\sigma(B) = 0$ and $\Lambda(B) > 0$. Then, by Lemma 4.2, there is a sequence (h_i), $i = 1, 2, \cdots$, in H such that $\sigma[X - \bigcup h_i(B)] = 0$. Since $\sigma(B) = 0$, $\sigma[h_i(B)] = 0$, $\sigma[\bigcup h_i(B)] = 0$ and $\sigma(X) = 0$. But σ is Haar with respect to H, hence $\sigma(X) > 0$. Thus if $\sigma(B) = 0$, $\Lambda(B) = 0$.

Assume $\Lambda(B) = 0$ and $\sigma(B) > 0$. Then, by Lemma 4.3, there is a sequence (h_i), $i = 1,2, \cdots$, in H such that $\sigma[X - \bigcup h_i(B)] = 0$, hence $\Lambda[X - \bigcup h_i(B)] = 0$. But since Λ is Haar with respect to H, $\Lambda(X) > 0$, and since $\Lambda[h_i(B)] = \Lambda(B)$, we have $\Lambda(B) > 0$, contradicting the assumption. Hence if $\Lambda(B) = 0$, then $\sigma(B) = 0$.

The separable metric space (X, d), the sigma-algebra of Borel sets $\mathcal{B}$, with either Λ or σ restricted to the Borel sets, form sigma-finite measure spaces, and by Lemma 4.4, Λ is absolutely continuous with respect to σ. Hence the Radon-Nikodym theorem may be applied [5, pp. 122–124], and there is a Radon-Nikodym derivative, a real-valued Borel-measurable function f on X, such that, for every Borel set B, $\Lambda(B) = \int_B f\, d\sigma$. In what follows, a Radon-Nikodym derivative f is fixed.

LEMMA 4.5. *For fixed t, $0 < t < \infty$, let $L_t = \{x\colon f(x) > t\}$, and let $S_t = \{x\colon f(x) < t\}$. Then at most one of L_t and S_t has positive σ-measure.*

PROOF. Assume $\sigma(L_t) > 0$. Then by Lemma 4.3 there is a sequence (h_i), $i = 1, 2, \cdots$, in H such that $\sigma[X - \bigcup h_i(L_t)] = 0$. For each i, $i = 1, 2, \cdots$, if $\sigma[S_t \cap h_i(L_t)] > 0$, we have

$$\Lambda[S_t \cap h_i(L_t)] = \int_{S_t \cap h_i(L_t)} f\, d\sigma < t\sigma[S_t \cap h_i(L_t)], \quad \text{and}$$

$$\Lambda[h_i^{-1}(S_t) \cap L_t] = \int_{h_i^{-1}(S_t) \cap L_t} f\, d\sigma > t\sigma[h_i^{-1}(S_t) \cap L_t],$$

contradicting the invariance of Λ and σ under H. Hence, if $\sigma(L_t) > 0$, then $\sigma(S_t) = 0$. In a similar fashion it can be shown that if $\sigma(S_t) > 0$, then $\sigma(L_t) = 0$.

LEMMA 4.6. *Let*

$$c = \sup \{t \colon \sigma\{x \colon f(x) \geqslant t\} > 0\}. \tag{7}$$

Then $0 < c < \infty$.

PROOF. If $c = 0$, then $\Lambda(X) = \int_X f\, d\sigma = 0$, which contradicts the fact that Λ is Haar with respect to H.

Assume $c = \infty$. Then for any Borel set B with $\Lambda(B) > 0$, we have $\sigma(B) > 0$, and by Lemma 4.5, for any t, $\sigma[B \cap \{x \colon f(x) \geqslant t\}] = \sigma(B)$. Hence $\Lambda(B) > t\sigma(B)$ for any t; thus for every Borel set B, $\Lambda(B) = \infty$. But since Λ is Haar, there is a Borel set of positive finite Λ-measure. Hence $0 < c < \infty$.

LEMMA 4.7. *For a.e.*(σ) x *in* X, $f(x) = c$.

PROOF. The set $\{x \colon f(x) > c\} = \bigcup \{x \colon f(x) > c + 1/n\}$, and $\sigma\{x \colon f(x) > c + 1/n\} = 0$; hence $\sigma\{x \colon f(x) > c\} = 0$.

The set $\{x \colon f(x) < c\} = \bigcup \{x \colon f(x) < c - 1/n\}$. By the definition of c, $\sigma\{x \colon f(x) > c - 1/n\} > 0$, for $n = 1, 2, \cdots$, hence by Lemma 4.5, $\sigma\{x \colon f(x) < c - 1/n\} = 0$ for every n, and $\sigma\{x \colon f(x) \neq c\} = 0$.

LEMMA 4.8. *For every Borel set* B, $\Lambda(B) = c\sigma(B)$.

PROOF. $\Lambda(B) = \int_B f\, d\sigma = \int_B c\, d\sigma = c\sigma(B)$.

THEOREM 4.9. *Let* (X, d) *be a separable metric space, let* H *be a family of homeomorphisms from* X *onto* X *which satisfies Condition* (I), *let* Λ *and* σ *be Borel-regular Carathéodory outer measures on* X *that are Haar with respect to* H, *and let* σ *satisfy the* 5-r *condition. Then there is a positive real number* c *such that for every* $E \subset X$, $\Lambda(E) = c\sigma(E)$.

PROOF. Fix $E \subset X$. Then since Λ and σ are Borel-regular, there are Borel sets B_1 and B_2 such that $E \subset B_1$, $E \subset B_2$, $\Lambda(E) = \Lambda(B_1)$, and $\sigma(E) = \sigma(B_2)$. Hence, with $B = B_1 \cap B_2$, B is a Borel set, $\Lambda(E) = \Lambda(B)$, and $\sigma(E) = \sigma(B)$.

By the previous lemma, with c defined as at (7), we have $\Lambda(E) = \Lambda(B) = c\sigma(B) = c\sigma(E)$.

The uniqueness theorem can now be proved.

PROOF OF THEOREM 1.7. By Theorem 2.1, if Λ_1 satisfies the 5-r condition at one point in X, then Λ_1 satisfies the 5-r condition on X. Letting Λ_1 be the σ of Theorem 4.9, with c defined as at (7), we have $\Lambda_2(E) = c\Lambda_1(E)$ for every $E \subset X$.

5. **Other criteria for uniqueness.** The density properties of a Haar measure Λ with respect to a 5-r measure σ may be used to decompose the space X into disjoint subsets. Letting

$$
\begin{aligned}
X_0 &= \left\{x\colon \liminf_{r\to 0} \frac{\Lambda c(x, r)}{\sigma c(x, r)} = 0\right\},\\
X_1 &= \left\{x\colon 0 < \liminf_{r\to 0} \frac{\Lambda c(x, r)}{\sigma c(x, r)} \leqslant \limsup_{r\to 0} \frac{\Lambda c(x, r)}{\sigma c(x, r)} < \infty\right\},\\
X_2 &= \left\{x\colon 0 < \liminf_{r\to 0} \frac{\Lambda c(x, r)}{\sigma c(x, r)} < \limsup_{r\to 0} \frac{\Lambda c(x, r)}{\sigma c(x, r)} = \infty\right\}, \quad \text{and}\\
X_\infty &= \left\{x\colon \liminf_{r\to 0} \frac{\Lambda c(x, r)}{\sigma c(x, r)} = \infty\right\},
\end{aligned} \tag{8}
$$

we have $X = X_0 \cup X_1 \cup X_2 \cup X_\infty$, and the sets are pairwise disjoint.

From Lemmas 3.3, 3.4, and 3.5, we have $\Lambda(X_0 \cup X_2) = 0$ and $\sigma(X_2 \cup X_\infty) = 0$.

If $X_1 = \varnothing$, or if X_1 is either Λ-null or σ-null, then there are disjoint sets A and B such that for every subset E of X,

$$\Lambda(E) = \Lambda(E \cap A), \quad \text{and} \quad \sigma(E) = \sigma(E \cap B),$$

where $A = X_\infty$ and $B = X_0$ if $X_1 = \varnothing$, or $B = X_0 \cup X_1$ if X_1 is Λ-null, and similarly if X_1 is σ-null. If this situation occurs, then both A and B are dense in X. This will be proved for the case $X_1 = \varnothing$; extension to the other cases is obvious.

THEOREM 5.1. *Assume* $X = X_0 \cup X_2 \cup X_\infty$ *as in* (8). *Then* $X = \operatorname{cl} X_0 = \operatorname{cl} X_\infty$, *where* $\operatorname{cl} E$ *means the closure of* E.

PROOF. Assume $X \neq \mathrm{cl}\, X_\infty$. Then there is an $x \in X$, and an open neighborhood O_x of x, such that $O_x \cap X_\infty = \emptyset$. Now $X = \bigcup\{h(O_x)\colon h \in H\}$, and since X is separable, $X = \bigcup h_i(O_x)$ for some sequence (h_i), $i = 1, 2, \cdots$, of sets in H, and $\Lambda(X) \leqslant \Sigma\Lambda[h_i(O_x)]$. Since $\Lambda[h_i(O_x)] = \Lambda(O_x) = \Lambda(O_x \cap X_\infty) = 0$, we have $\Lambda(X) = 0$, which is a contradiction, since Λ is Haar. Hence $X = \mathrm{cl}\, X_\infty$. Similarly $X = \mathrm{cl}\, X_0$, since σ satisfies the 5-r condition, hence is positive on every open set.

If the set $X_1 \neq \emptyset$, then there is a point x in X at which the Haar measure Λ satisfies the 5-r condition, as is shown in the following theorem.

THEOREM 5.2. *Let (X, d) be a separable metric space, let σ be a Borel-regular Carathéodory outer measure in X which satisfies the 5-r condition, let Λ be a locally-finite Borel-regular Carathéodory outer measure in X, and let x be a point in X such that*

$$0 < \liminf_{r\to 0} \frac{\Lambda[c(x, r)]}{\sigma[c(x, r)]} \leqslant \limsup_{r\to 0} \frac{\Lambda[c(x, r)]}{\sigma[c(x, r)]} < \infty.$$

Then Λ satisfies the 5-r condition at x.

PROOF. There are real numbers a and b such that

$$0 < a = \liminf_{r\to 0} \frac{\Lambda[c(x, r)]}{\sigma[c(x, r)]} \leqslant \limsup_{r\to 0} \frac{\Lambda[c(x, r)]}{\sigma[c(x, r)]} = b < \infty.$$

Fix $\epsilon > 0$ with $\epsilon < a$. Then there is an r_0 such that for $0 < r < r_0$

$$a - \epsilon < \frac{\Lambda[c(x, r)]}{\sigma[c(x, r)]} < b + \epsilon.$$

Since σ satisfies the 5-r condition, there are k and K such that $\sigma[c(x, 5r)] < K\sigma[c(x, r)]$ for $0 < r < k$. Then for $0 < r < r_0/5, r < k$, we have

$$\begin{aligned}\Lambda[c(x, 5r)] &< (b + \epsilon)\sigma[c(x, 5r)] < (b + \epsilon)K\sigma[c(x, r)] \\ &< \frac{b + \epsilon}{a - \epsilon} K\Lambda[c(x, r)],\end{aligned}$$

and Λ satisfies the 5-r condition at x.

There is also a condition on a point in X_∞ which ensures that Λ satisfies the 5-r condition.

THEOREM 5.3. *Let (X, d) be a separable metric space, let σ be a Borel-regular Carathéodory outer measure in X which satisfies the 5-r condition, let Λ be a locally-finite Borel-regular Carathéodory outer measure in X, and let x be a point in X such that*

(1) $\liminf_{r\to 0} \Lambda[c(x, r)]/\sigma[c(x, r)] = \infty$, *and*

(2) *there are positive constants* $r_0, m < 1$ *such that, for* $0 < r < r_0$ *and for* $0 < t < mr$,

$$\frac{\Lambda[c(x, t)]}{\sigma[c(x, t)]} \geqslant \frac{\Lambda[c(x, r)]}{\sigma[c(x, r)]}.$$

Then Λ *satisfies the* 5-*r* *condition at* x.

PROOF. Fix x, where x satisfies (1) and (2). Then $x \in A_n$, for some n, where A_n is defined as in (2), and, for $0 < r < 1/n$, $\sigma[c(x, 5r)] < n\sigma[c(x, r)]$. For the m in (2), there is a positive integer t with $5^{-t} < m$. Then, for $0 < r < 1/n$, $r < r_0/5$, we have

$$\frac{\Lambda[c(x, 5r)]}{\sigma[c(x, 5r)]} \leqslant \frac{\Lambda[c(x, rm)]}{\sigma[c(x, rm)]} \leqslant \frac{\Lambda[c(x, r)]}{\sigma[c(x, r5^{-t})]} < \frac{n^{t+1}\Lambda[c(x, r)]}{\sigma[c(x, 5r)]}$$

and Λ satisfies a 5-*r* condition at x.

An additional condition imposed on the 5-*r* measure σ will ensure that a Haar measure Λ satisfies the 5-*r* condition.

LEMMA 5.4. *Let* (X, d) *be a separable metric space, let* H *be a family of homeomorphisms from* X *onto* X *which satisfies Condition* (I), *let* Λ *be a Borel-regular Carathéodory outer measure that is Haar with respect to* H, *and let* σ *be a Borel-regular Carathéodory outer measure that satisfies the* 5-*r* *condition and has the property that for* x, y *in* X *there are positive real numbers* m *and* M *such that for* $0 < r < m$,

$$\sigma[c(x, r)] < M\sigma[c(y, r)], \quad \textit{and} \quad \sigma[c(y, r)] < M\sigma[c(x, r)].$$

Then there is a point x *in* X *such that* Λ *satisfies the* 5-*r* *condition at* x.

PROOF. If $X_1 \neq \emptyset$, with X_1 defined as in (8), then by Lemma 5.2, Λ satisfies the 5-*r* condition on X.

Assume $X_1 = \emptyset$. Then $X_\infty \neq \emptyset$ and $X_0 \neq \emptyset$, by Theorem 5.1. For $x \in X_\infty$, $y \in X_0$, we have m and M as above. Note that for $0 < r < m$,

$$\frac{1}{M}\sigma[c(x, r)] < \sigma[c(y, r)] < M\sigma[c(x, r)],$$

and

$$\frac{1}{M}\sigma[c(y, r)] < \sigma[c(x, r)] < M\sigma[c(y, r)].$$

Since σ satisfies the 5-*r* condition, there is an n such that if $0 < r < 1/n$, $\sigma[c(y, 5r)] < n\sigma[c(y, r)]$.

Since H satisfies Condition (I), there are $l(x, y), L(x, y)$ and an h in H such that for $0 < r < l(x, y)$, we have $h(y) = x$ and $c(x, r) \subset h[c(y, rL)]$. There is a positive integer q such that $5^{q-1} \leqslant L(x, y) < 5^q$.

Since $\liminf_{r\to 0} \Lambda[c(y, r)]/\sigma[c(y, r)] = 0$, there is a sequence (r_i), $i = 1, 2, \cdots$, $\lim_{i\to\infty} r_i = 0$, $r_{i+1} < r_i$, and $\lim_{i\to\infty} \Lambda[c(y, r_i)]/\sigma[c(y, r_i)] = 0$. We may assume

$$\frac{\Lambda[c(y, r_i)]}{\sigma[c(y, r_i)]} < 1 \quad \text{for } i = 1, 2, \cdots .$$

Let $t > Mn^q$. Then, since $\lim_{r\to 0} \Lambda[c(x, r)]/\sigma[c(x, r)] = \infty$, there is an r_0 such that, for $0 < r < r_0$,

$$\frac{\Lambda[c(x, r)]}{\sigma[c(x, r)]} > t > Mn^q.$$

Fix r with $0 < r < r_0$, $r < 1/n$, $r < l(x, y)$, $r < m$, and

$$\Lambda[c(y, r)]/\sigma[c(y, r)] < 1.$$

Then

$$\begin{aligned}\Lambda[c(x, r5^{-q})] &> t\sigma[c(x, r5^{-q})] > Mn^q\sigma[c(x, r5^{-q})] > M\sigma[c(x, r)]\\ &> \sigma[c(y, r)] > \Lambda[c(y, r)] = \Lambda\{h[c(y, r)]\} \geqslant \Lambda[c(x, r/L)]\\ &\geqslant \Lambda[c(x, r5^{-q})];\end{aligned}$$

a contradiction. Hence, if $X_\infty \neq \emptyset$, then $X_0 = \emptyset$, and since σ is positive on X and null on $X_\infty \cup X_2$, we have $X_1 \neq \emptyset$, contradicting the assumption that $X_1 = \emptyset$. Since $X_1 \neq \emptyset$, there is a point $x \in X_1$, and by Lemma 5.2, Λ satisfies the 5-r condition at x.

The preceding lemmas are summarized in the following theorem.

THEOREM 5.5. *Let (X, d) be a separable metric space, let H be a family of homeomorphisms from X onto X which satisfies Condition* (I), *let Λ and Λ_1 be a Borel-regular Carathéodory outer measures on X that are Haar with respect to H, and let σ be a Borel-regular Carathéodory outer measure on X which satisfies the 5-r condition. Then if* (i), (ii), *or* (iii) *holds, there is a positive number c such that for every $E \subset X$, $\Lambda(E) = c\Lambda_1(E)$.*

(i) *There is a point x in X such that*

$$0 < \liminf_{r\to 0} \frac{\Lambda[c(x, r)]}{\sigma[c(x, r)]} \leqslant \limsup_{r\to 0} \frac{\Lambda[c(x, r)]}{\sigma[c(x, r)]} < \infty.$$

(ii) *There is a point x in X such that*

$$\liminf_{r\to 0} \frac{\Lambda[c(x, r)]}{\sigma[c(x, r)]} = \infty,$$

and there are positive constants r_0 and $m < 1$ such that for $0 < r < r_0$ and $0 < t < mr$,

$$\frac{\Lambda[c(x, t)]}{\sigma[c(x, t)]} \geqslant \frac{\Lambda[c(x, r)]}{\sigma[c(x, r)]}.$$

(iii) *For every pair of points* x, y *in* X *there are positive real numbers* m *and* M *such that for* $0 < r < m$,

$$\sigma[c(x, r)] < M\sigma[c(y, r)], \quad \textit{and} \quad \sigma[c(y, r)] < M\sigma[c(x, r)].$$

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